March 4, 2009 WOCOMAL Varsity Meet

Round 2: Algebra I

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Find all the 2-digit whole numbers that exactly divide 2^{40}.

2. Solve: $\frac{60}{y^2-36}+1=\frac{5}{y-6}$.

3. Over the set of real numbers find the domain of $\frac{f(x)}{g(x)}$, if $f(x)=\sqrt{x+2}$ and $g(x)=\frac{x}{x-6}$.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Doherty, St. Peter-Marian, Southbridge

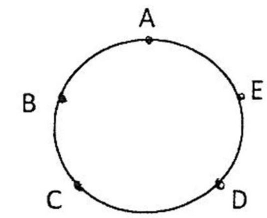
March 4, 2009 WOCOMAL Varsity Meet

Round 3: Open Geometry

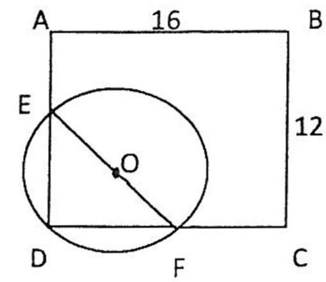
ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

EXPRESS ALL ANSWERS IN TERMS OF π.

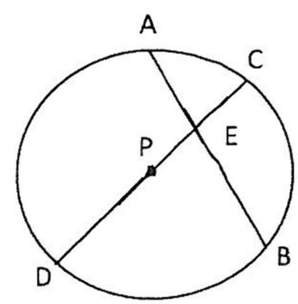
1. In the figure A, B, C, D, and E divide the circle into 5 equal arcs. If the area of the circle is 64π, what is the length of arc ABC?



2. ABCD is a rectangle. E is the midpoint of $\overline{AD}$ and F is the midpoint of $\overline{CD}$. $\overline{EF}$ is the diameter of circle O. If AB=16 BC=12, find the area of circle O.



3. In circle P, chord $\overline{AB}$ intersects diameter $\overline{CD}$ at E and PE=3. If AE=5, and EB=5.4, what is the area of circle P?



ANSWERS (Express all answers in terms of π.)

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Assabet Valley, Bancroft, Hudson

March 4, 2009 WOCOMAL Varsity Meet

Round 4: Logs, Exponents, and Radicals

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Simplify: $(9^{\frac{1}{2}} + (16)^{\frac{1}{2}})^2$.

2. Simplify: $\log_{0.2}\sqrt{5}$.

3. Find all ordered pairs (x, y) such that:

$$x + 2y + \sqrt{2xy} = 24 \text{ and}$$
$$x^2 + 4y^2 + 2xy = 192.$$

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Auburn, Shepherd Hill, Shrewsbury

March 4, 2009 WOCOMAL Varsity Meet

Round 5: Trigonometry

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Find the simplified numerical value of the following quotient:

$$\frac{\sin 10^o \cos 10^o \tan 10^o \cot 10^o \sec 10^o \csc 10^o}{\sin 20^o \cos 20^o \tan 20^o \cot 20^o \sec 20^o \csc 20^o}.$$

2. Evaluate: $\dfrac{\tan\dfrac{5\pi}{4} - \cos\dfrac{5\pi}{3}}{5\sin\dfrac{\pi}{6}}$.

3. Solve for all x where $0 \le x < 2\pi$. Leave all answer(s) in terms of π.

$$2\sin^2 x\cos^2 x + \sin x\cos^2 x - 2\sin^2 x - \sin x = 0.$$

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Bartlett, Burncoat, Clinton

March 4, 2009 TEAM ROUND ANSWER SHEET WOCOMAL Varsity Meet

(2 points each)

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________ feet

5. ______________________________

6. ______________________________

7. ______________________________

8. ______________________________

9. ______________________________

School ______________________________

Team # _____________________________

Student Names:

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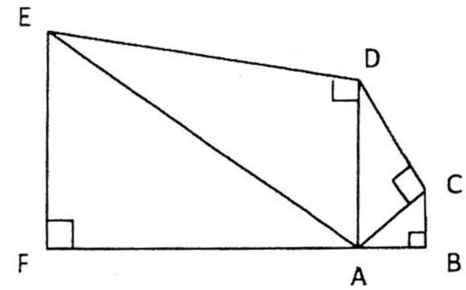
ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM AND ON THE SEPARATE TEAM ANSWER SHEET (2 points each)

1. Find the least integer x for which $\frac{12}{x+1}$ represents an integer.

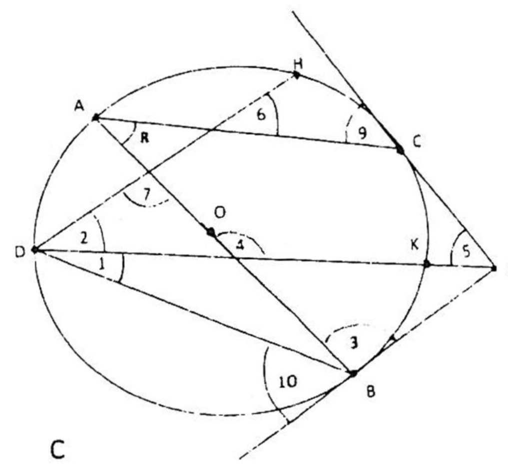


2. Simplify: $\frac{2}{2x+3} \div \left(\frac{1}{x-1} - \frac{2x-5}{2x^2+x-3}\right)$.

3. Solve for x: $\log_{27}(\log_8 512) = \log_5 x$.

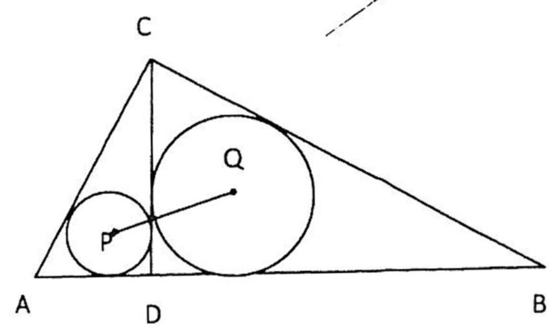
4. Given the four right triangles as shown on the right. $\overline{FB}$ is a straight line segment. $\measuredangle CAB = 30^\circ$, $\measuredangle CAD = 45^\circ$, and $\measuredangle DAE = 60^\circ$. If BC=100 feet, find AF rounded to the nearest foot.

5. Find all real numbers x such that: $\sqrt[3]{5+x} + \sqrt[3]{5-x} = 1$.



6. Find all values of x for which $3^{4x} + 27 = 4(3^{2x+1})$.

7. $\overrightarrow{FC}$ and $\overrightarrow{FB}$ are tangent to circle O. $\widehat{AD} = 60^o$, $\widehat{AH} = 50^o$, $\widehat{HC} = 30^o$, $\widehat{BK} = 50^o$. Find the sum of the measures of: $\measuredangle 1 + \measuredangle 2 + \measuredangle 3 + \measuredangle 4 + \measuredangle 5 + \measuredangle 6 + \measuredangle 7 + \measuredangle 8 + \measuredangle 9 + \measuredangle 10$.



8. In triangle ABC, $\measuredangle C$ is a right angle. $\overline{CD}$ is an altitude. The circles centered at P and Q are inscribed in $\triangle ADC$ and $\triangle CDB$ respectively. If AC=15 and BC=20, find the length of PQ.

9. A "happy" number is defined by the following process. Starting with any positive integer, replace the number by the sum of the squares of its digits, and repeat the process until the number equals 1. (For example: 23 is a happy number. The associated sequence is 23; $2^2 + 3^2 = 13$; $1^2 + 3^2 = 10$; $1^2 + 0^2 = 1$.) Find all the happy numbers between 90 and 99 inclusive.

Assabet Valley, Auburn, Burncoat, Hudson, Notre Dame, Quaboag, St. John's, Westborough

Round 1: Elementary Number Theory

(1 pt.) 19 (tomato plants)

(2 pts.) 156

(3 pts.) 503_7 or 503 (accept either answer)

Round 2: Algebra I

(1 pt.) 16, 32, 64 (need all 3 answers, any order)

(2 pts.) -1 (6 is extraneous)

(3 pts.) $x \geq -2$, but $x \neq 0$ or $x \neq 6$ (or any equivalent form)

Round 3: Open Geometry

(All answers must be written in terms of π.)

(1 pt.) $\frac{32\pi}{5}$ or 6.4π or $6\frac{2}{5}\pi$

(2 pts.) 25π

(3 pts.) 36π

Round 4: Logs, Exponents, and Radicals

(1 pt.) 49

(2 pts.) $-\frac{1}{2}$ or -0.5

(3 pts.) (8, 4) (must include parenthesis)

Round 5: Trigonometry

(1 pt.) 1

(2 pts.) $\frac{1}{5}$ or 0.2

(3 pts.) $0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$ (need all 4 answers, written in terms of π, any order.)

March 4, 2009 TEAM ROUND ANSWERS WOCOMAL Varsity Meet

(2 points each)

1. -13

2. $\frac{x-1}{4}$ or $\frac{1}{4}(x-1)$

3. $5^{\frac{1}{3}}$ or $\sqrt[3]{5}$ or 1.710

4. 400 feet

5. $\pm 2\sqrt{13}$ or ± 7.211

6. 1, $\frac{1}{2}$ or 0.5 (need both answers)

7. 605 (with or without degree sign)

8. $5\sqrt{2}$ or 7.071

9. 91, 94, 97 (need all 3 answers, in any order)

------------------- *Elementary Number Theory* --------------------

1. If you plant 6 to a row and have one left over means the number of tomato plants is one more than a multiple of 6 (7, 13, 19, 25, 31, 37, 43). Now divide these possibilities by 5 and see which ones give a remainder of 4. The answer is $\boxed{19}$ tomato plants.

2. $((8\,12)\ [6\ 10]) \Rightarrow (24-4\ \ 30+2) \Rightarrow (20\ \ 32) \Rightarrow 160-4=\boxed{156}$.

3. $xyz_7 = zyx_9$ where $x,\ y,\ z \in 0, 1, 2, 3, 4, 5,$ or 6. Hence $49x+7y+z=81z+9y+x$. Rearranging: $48x-2y-80z=0 \Rightarrow 24x-y-40z=0 \Rightarrow y=24x-40z$ or $y=8(3x-5z)$. The only value for y is 0. Which implies $3x-5z=0$ or $3x=5z \Rightarrow x=5$ and $z=3$. Therefore the number is $\boxed{503_7}$.

------------------- *Algebra I* --------------------

1. $2^4=16,\ 2^5=32,\ 2^6=64$. Answer: $\boxed{16, 32, 64}$.

2. Multiply each term by y^2-36 to get $60+y^2-36=5y+30$. Rearrange to get $y^2-5y+6=0$. Factor and solve (or use the quadratic formula) to get $y=-1,6$. Because of the denominator $y \neq 6$, therefore $y=\boxed{-1}$.

3. $\dfrac{\text{f(x)}}{\text{g(x)}} = \dfrac{\sqrt{x+2}}{\dfrac{x}{x-6}}$, here x ≠ 6. $\dfrac{(x-6)\sqrt{x+2}}{x}$, here $x \neq 0$. Since $x+2 \geq 0,\ x \geq -2$.
   Answer $\boxed{x \geq -2, \text{ but } x \neq 0 \text{ or } x \neq 6}$.

-------------------- *Open Geometry* --------------------

1. The radius of the circle is 8. $\overset{\frown}{ABC} = \left(\dfrac{\frac{2}{5}(360)}{360}\right)(16\pi) = \boxed{\dfrac{32\pi}{5} \text{ or } 6.4\pi \text{ or } 6\dfrac{2}{5}\pi}$.

2. ED=6 and DF=8. By the Pythagorean theorem EF=10. Hence the radius of the circle is 5 and the area=$\boxed{25\pi}$.

-------------------- *Open Geometry (continued)* --------------------

3. $(AE)(EB)=(DE)(EC)$ where $DE=DP+PE$ and $EC=PC-PE$. Therefore $(5)(5.4)=(R+3)(R-3)$ or $27=R^2-9 \Rightarrow R^2=36$ or $R=6$. Thus the area of the circle is $\boxed{36\pi}$.

-------------------- *Logs, Exponents, and Radicals* --------------------

1. $\left(9^{\frac{1}{2}}+(16)^{\frac{1}{2}}\right)^2=(3+4)^2=7^2=\boxed{49}$.

2. $\log_{0.2}\sqrt{5}=\dfrac{\ln\sqrt{5}}{\ln\frac{1}{5}}=\dfrac{\ln 5^{\frac{1}{2}}}{\ln 5^{-1}}=\dfrac{\frac{1}{2}\ln 5}{-1\ln 5}=\boxed{-\frac{1}{2}\text{ or }-0.5}$.

3. Rearrange the first equation: $x+2y=24-\sqrt{2xy}$.
Square both sides: $x^2+4xy+4y^2=576-48\sqrt{2xy}+2xy$ or $x^2+2xy+4y^2=576-48\sqrt{2xy}$.
Substituting with the second equation: $192=576-48\sqrt{2xy}$ or $48\sqrt{2xy}=384 \Rightarrow \sqrt{2xy}=8$ or $2xy=64$.
Substitute $\sqrt{2xy}=8$ or $2xy=64$ into the original equations: $x+2y=16$ and $x^2+4xy^2=128$.
Rearrange $x+2y=16$ as $x=16-2y$ and substitute into $x^2+4xy^2=128$: $256-64y+4y^2+4y^2=128$.
Rearrange and solve for y: $256-64y+4y^2+4y^2=128 \Rightarrow y^2-8y+16=0$ or $y=4$.
Since $x=16-2y$, $x=8$. Hence $\boxed{(8,\ 4)}$.

-------------------- *Trigonometry* --------------------

1. Substituting with reciprocals: $\dfrac{\sin 10^\circ \cos 10^\circ \tan 10^\circ \dfrac{1}{\tan 10^\circ}\dfrac{1}{\cos 10^\circ}\dfrac{1}{\sin 10^\circ}}{\sin 20^\circ \cos 20^\circ \tan 20^\circ \dfrac{1}{\tan 20^\circ}\dfrac{1}{\cos 20^\circ}\dfrac{1}{\sin 20^\circ}}=\boxed{1}$.

2. $\dfrac{\tan\frac{5\pi}{4}-\cos\frac{5\pi}{3}}{5\sin\frac{\pi}{6}}=\dfrac{\tan\frac{\pi}{4}-\cos\frac{\pi}{3}}{5\sin\frac{\pi}{6}}=\dfrac{1-\frac{1}{2}}{5\left(\frac{1}{2}\right)}=\dfrac{\frac{1}{2}}{5\left(\frac{1}{2}\right)}=\boxed{\frac{1}{5}\text{ or }0.2}$.

------------------ *Trigonometry (continued)* ------------------

3. $\sin x(2\sin x\cos^2 x+\cos^2 x-2\sin x-\sin x)=0 \Rightarrow \sin x[\cos^2 x(2\sin x+1)-1(2\sin x+1)]=0 \Rightarrow$

$\sin x(2\sin x+1)(\cos x+1)(\cos x-1)=0 \Rightarrow \sin x=0,\ \sin x=-\frac{1}{2},\ \cos x=-1, \text{ or } \cos x=1.$

$\Rightarrow \boxed{0,\ \pi,\ \frac{7\pi}{6},\ \frac{11\pi}{6}}$.

-------------------- *Team* --------------------

1. For $\frac{12}{x+1}$ to be an integer, $x+1$ must be an integral factor of 12. The least integral factor of 12 is -12. Therefore $x+1=-12 \Rightarrow x=\boxed{-13}$.

2. $\frac{2}{2x+3} \div \left(\frac{1(2x+3)-(2x-5)}{(x-1)(2x+3)}\right)=\frac{2}{2x+3}\left(\frac{(x-1)(2x+3)}{4}\right)=\boxed{\frac{x-1}{4} \text{ or } \frac{1}{4}(x-1)}$.

3. $\log_{27}(\log_8 512)=\log_5 x \Rightarrow \log_{27} 3=\log_5 x \Rightarrow \frac{1}{3}=\log_5 x \Rightarrow x=\boxed{5^{\frac{1}{3}} \text{ or } \sqrt[3]{5} \text{ or } 1.710}$.

4. Let $AC=x$, $AD=y$, $AE=z$, and $AF=Q$. Then $\sin 30^o=\frac{100}{x} \Rightarrow x=\frac{100}{\sin 30^o}$, $\cos 45^o=\frac{x}{y} \Rightarrow$ $y=\frac{x}{\cos 45^o}$, $\cos 60^o=\frac{y}{z} \Rightarrow z=\frac{y}{\cos 60^o}$, $\cos 45^o=\frac{Q}{z} \Rightarrow Q=z\cos 45^o$. (Note each of these equations can be evaluated and used to solve the next equation.) But by combining all the equations

$$\Rightarrow Q=z\cos 45^o=\left(\frac{y}{\cos 60^o}\right)\cos 45^o=\left(\frac{\frac{x}{\cos 45^o}}{\cos 60^o}\right)\cos 45^o=\frac{x}{\cos 60^o}=\frac{\frac{100}{\sin 30^o}}{\cos 60^o}=\boxed{400} \text{ feet.}$$

5. Let $A=\sqrt[3]{5+x}+\sqrt[3]{5-x}=1$. Then $A^3=5+x+5-x+3(\underline{\sqrt[3]{5+x}+\sqrt[3]{5-x}})(\sqrt[3]{5+x}\sqrt[3]{5-x})=1 \Rightarrow$ $10+3(1)\sqrt[3]{25-x^2}=1 \Rightarrow 3\sqrt[3]{25-x^2}=-9 \Rightarrow \sqrt[3]{25-x^2}=-3 \Rightarrow 25-x^2=-27 \Rightarrow x^2=52 \Rightarrow x=\pm\sqrt{52}$

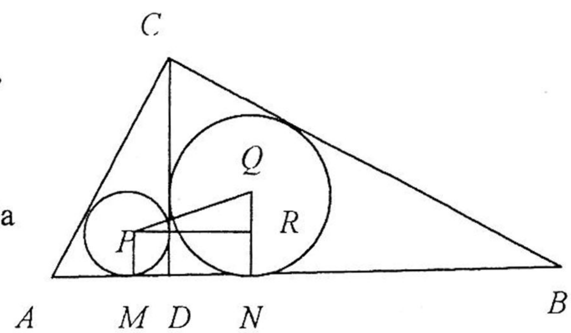
$\Rightarrow x=\boxed{\pm 2\sqrt{13} \text{ or } \pm 7.211}$.

--------------------- *Team (continued)* ---------------------

6. $3^{4x}+27=4(3^{2x+1})\Rightarrow 3^{4x}-4(3^{2x+1})+27=0\Rightarrow 3^{4x}-4(3)(3^{2x})+27=0\Rightarrow 3^{4x}-12(3^{2x})+27\Rightarrow$ $(3^{2x}-9)(3^{2x}-3)=0\Rightarrow(3^{2x}-3^2)(3^{2x}-3^1)=0\Rightarrow 3^{2x}-3^2=0$ or $3^{2x}-3^1=0\Rightarrow 2x=2$ or $2x=1$ $\Rightarrow x=\boxed{1,\ \frac{1}{2}\text{ or }0.5}$.

7. Since $\overline{AB}$ is a diameter, $\widehat{CK}=50^\circ$ and $\widehat{DB}=120^\circ$. Using the relationships between arcs and central angles , inscribed angles, and angles formed by lines intersecting in the exterior or interior of a circle: $\measuredangle 1=25^o$, $\measuredangle 2=40^o$, $\measuredangle 3=90^o$, $\measuredangle 4=125^o$, $\measuredangle 5=45^o$, $\measuredangle 6=45^o$, $\measuredangle 7=85^o$, $\measuredangle 8=50^o$, $\measuredangle 9=40^o$, $\measuredangle 10=60^o$. Hence $\measuredangle 1+\measuredangle 2+\measuredangle 3+\measuredangle 4+\measuredangle 5+\measuredangle 6+\measuredangle 7+\measuredangle 8+\measuredangle 9+\measuredangle 10=25^o+40^o+90^o+125^o$ $+45^o+45^o+85^o+50^o+40^o+60^o=\boxed{605^o}$.

8. Since $\triangle$ABC is a right triangle with AC=15 and BC=20, Then AB=25. Now (DC)(AB)=(AC)(BC) or DC(25)= =15(20)$\Rightarrow$ DC=12. Since $\triangle$ADC is a right triangle AC =15 and DC=12 then AD=9 and thus BC=16. A formula for the area of a triangle is half the radius times the perimeter or A=$\frac{1}{2}$rp. In $\triangle ADC$ $54=\frac{1}{2}r(36)$ or r=3



and in $\triangle$BCD 96=$\frac{1}{2}$r(48) or r=4. Draw perpendiculars from P to AD at M and from Q to BD at N making MN=7. Connecting a parallel line segment P to QN at R, makes PRNM a rectangle and $\triangle$PRQ a right triangle. Thus PR=7. Since RN=3, then RQ=1. Therefore PQ=$\sqrt{50}$=$\boxed{5\sqrt{2}}$ or $\boxed{7.071}$.

9. The happy numbers between 90 and 99 inclusive are: $\boxed{91, 94, \text{and } 97}$. The associated sequence for the happy numbers:
$91;\ 9^2+1^2=82;\ 8^2+2^2=68;\ 6^2+8^2=100;\ 1^2+0^2+0^2=1.$
$94;\ 9^2+4^2=\underline{97};\ 9^2+7^2=130;\ 1^2+3^2+0^2=10;\ 1^2+0^2=1.$
$97;\ 9^2+7^2=130;\ 1^2+3^2+0^2=10;\ 1^2+0^2=1.$